Probabilistic couplings for cryptography and privacy

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Relational properties

Properties about two runs of the same program

- Assume inputs are related by Ψ
- Want to prove the outputs are related by Φ

Examples

Monotonicity

- $\Psi : in_1 \leq in_2$
- $\Phi : out_1 \leq out_2$
- "Bigger inputs give bigger outputs"

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Non-interference

- $\Psi : lowinp_1 = lowinp_2$
- $\Phi : lowout_1 = lowout_2$
- "If low inputs are equal, then low outputs are equal"

Probabilistic relational properties

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Richer properties

- Indistinguishability, differential privacy

Probabilistic couplings

- Used by mathematicians for proving relational properties
- Applications: Markov chains, probabilistic processes

Idea

- Place two processes in the same probability space
- Coordinate the sampling

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Why is this interesting?

- Proving relational probabilistic properties reduced to proving non-relational non-probabilistic properties
- Compositional

Introducing probabilistic couplings

Basic ingredients

- Given: two distributions X_1, X_2 over set A
- Produce: joint distribution Y over $A \times A$
  - Projection over the first component is X_1
  - Projection over the second component is X_2

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Definition

Given two distributions X_1, X_2 over a set A, a coupling Y is a distribution over $A \times A$ such that $\pi_1(Y) = X_1$ and $\pi_2(Y) = X_2$

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$$\pi_1(Y)(a_1) = \sum_{a_2} Y(a_1, a_2)$$

Fair coin toss

- One way to coordinate: require $x_1 = x_2$
- A different way: require $x_1 = \neg x_2$
- Yet another way: product distribution
- Choice of coupling depends on application
- Couplings always exist

Couplings vs liftings

Let $\mu_1, \mu_2 \in \mathsf{Distr}(A)$, $\mu \in \mathsf{Distr}(A \times A)$ and $R \subseteq A \times A$. Then

$$\mu \blacktriangleleft_R \langle \mu_1 \ \& \ \mu_2 \rangle \triangleq \pi_1(\mu) = \mu_1 \wedge \pi_2(\mu) = \mu_2 \wedge \Pr_{y \leftarrow \mu}[y \in R] = 1$$

Different couplings yield liftings for different relations

Convergence of random walks

Simple random walk on integers

- Start at some position p
- Each step, flip coin $x \xleftarrow{\$} \mathit{flip}$
- Heads: $p \leftarrow p + 1$
- Tails: $p \leftarrow p - 1$

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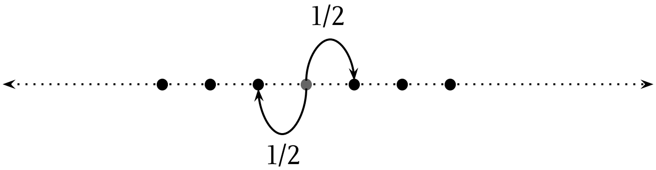

Coupling the walks to meet

Case $p_1 = p_2$: Walks have met

- Arrange samplings $x_1 = x_2$
- Continue to have $p_1 = p_2$

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Under coupling, if walks meet, they move together

Why is this interesting?

Memorylessness

Positions converge as we take more steps

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Coupling bounds distance between distributions

- Once walks meet, they stay equal
- Distance is at most probability walks don't meet

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Theorem

If Y is a coupling of two distributions (X_1, X_2), then

$$\|X_1 - X_2\|_{TV} \triangleq \sum_{a \in A} |X_1(a) - X_2(a)| \leq \Pr_{(y_1, y_2) \sim Y}[y_1 \neq y_2].$$

probabilistic Relational Hoare Logic

$\vdash \{P\} c_1 \sim c_2 \{Q\}$ iff there exists μ such that

$$P(m_1 \uplus m_2) \Rightarrow \mu \blacktriangleleft_Q \langle [\![c_1]\!]\ m_1\ \&\ [\![c_2]\!]\ m_2 \rangle$$

where

$$\mu \blacktriangleleft_R \langle \mu_1\ \&\ \mu_2 \rangle \triangleq \pi_1(\mu) = \mu_1 \wedge \pi_2(\mu) = \mu_2 \wedge \mathrm{supp}(\mu) \subseteq R$$

Fundamental lemma of pRHL

If $Q \triangleq E_1 \Rightarrow E_2$ then $\Pr_{([\![c_1]\!] m_1)}[E_1] \leq \Pr_{([\![c_2]\!] m_2)}[E_2]$

Core rules

$$\frac{\{\Phi\}c_1 \sim c_2\{\Theta\} \quad \{\Theta\}c_1' \sim c_2'\{\Psi\}}{\{\Phi\}c_1; c_1' \sim c_2; c_2'\{\Psi\}}$$

$$\frac{\{\Phi \wedge b_1 \wedge b_2\}c_1 \sim c_2\{\Psi\} \quad \{\Phi \wedge \neg b_1 \wedge \neg b_2\}c_1' \sim c_2'\{\Psi\}}{\{\Phi \wedge b_1 = b_2\}\text{if } b_1 \text{ then } c_1 \text{ else } c_1' \sim \text{if } b_2 \text{ then } c_2 \text{ else } c_2'\{\Psi\}}$$

$$\frac{\{\Phi \wedge b_1 \wedge b_2\}c_1 \sim c_2\{\Phi \wedge b_1 = b_2\}}{\{\Phi \wedge b_1 = b_2\}\text{while } b_1 \text{ do } c_1 \sim \text{while } b_2 \text{ do } c_2\{\Phi \wedge \neg b_1 \wedge \neg b_2\}}$$

Loops

$$\dfrac{\begin{array}{c}\Psi \implies p_0 \oplus p_1 \oplus p_2 \\ \Psi \wedge p_0 \implies e_1 \wedge e_2 \qquad \Psi \wedge p_1 \implies e_1 \qquad \Psi \wedge p_2 \implies e_2 \\ \mathsf{while}\ e_1 \wedge p_1\ \mathsf{do}\ c_1 \Downarrow \mathsf{while}\ e_2 \wedge p_2\ \mathsf{do}\ c_2 \\ \{\Psi \wedge p_1\} c_1 \sim \mathsf{skip}\{\Psi\} \qquad \{\Psi \wedge p_2\}\mathsf{skip} \sim c_2\{\Psi\} \\ \{\Psi \wedge p_0\} c_1 \sim c_2\{\Psi\}\end{array}}{\{\Psi\}\mathsf{while}\ e_1\ \mathsf{do}\ c_1 \sim \mathsf{while}\ e_2\ \mathsf{do}\ c_2\{\Psi \wedge \neg e_1 \wedge \neg e_2\}}$$

Random assignment

$$\frac{\mu \blacktriangleleft_Q \langle \mu_1 \,\&\, \mu_2 \rangle}{\vdash \{\top\} x_1 \xleftarrow{\$} \mu_1 \sim x_2 \xleftarrow{\$} \mu_2 \{Q\}}$$

Specialized rule

$$\frac{f \in T \xrightarrow{1-1} T \qquad \forall v \in T.\, d_1(v) = d_2(f\ v)}{\vdash \{\forall v, Q[v/x_1, f\ v/x_2]\} x_1 \xleftarrow{\$} \mu_1 \sim x_2 \xleftarrow{\$} \mu_2 \{Q\}}$$

Notes

- Bijection f: specifies how to coordinate the samples
- Side condition: marginals are preserved under f
- Assume: samples coupled when proving postcondition Φ

Applications to cryptography

- EasyCrypt: interactive proof assistant (inspired from ssreflect) with back-end to SMT and CAS
- applied to encryption, signatures, hash designs, key exchange protocols, zero knowledge protocols, garbled circuits, SHA3, e-voting

Formalizing cryptographic proofs?

- *In our opinion, many proofs in cryptography have become essentially unverifiable. Our field may be approaching a crisis of rigor*. Bellare and Rogaway, 2004-2006
- *Do we have a problem with cryptographic proofs? Yes, we do [...] We generate more proofs than we carefully verify (and as a consequence some of our published proofs are incorrect)*. Halevi, 2005

approximate probabilistic Relational Hoare Logic

- Quantitative generalization of pRHL $\vdash_{\epsilon,\delta} \{P\} c_1 \sim c_2 \{Q\}$
- Valid if there exists μ_L, μ_R such that

$$P(m_1 \uplus m_2) \implies \mu_L, \mu_R \blacktriangleleft_Q^{\epsilon,\delta} \langle [\![c_1]\!]\ m_1 \ \&\ [\![c_2]\!]\ m_2 \rangle$$

where

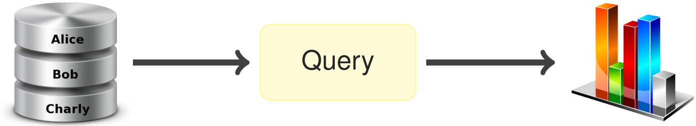
$$\mu_L, \mu_R \blacktriangleleft_Q^{\epsilon,\delta} \langle \mu_1 \ \&\ \mu_2 \rangle \triangleq \begin{cases} \pi_1(\mu_L) = \mu_1 \wedge \pi_2(\mu_R) = \mu_2 \\ \mathrm{supp}(\mu_L), \mathrm{supp}(\mu_R) \subseteq Q \\ \Delta_\epsilon(\mu_1, \mu_2) \leq \delta \end{cases}$$

- Fundamental theorem of apRHL: if $Q \triangleq E_1 \Rightarrow E_2$ then

$$\Pr_{([\![c_1]\!]\ m_1)}[E_1] \leq \exp(\epsilon) \Pr_{([\![c_2]\!]\ m_2)}[E_2] + \delta$$

- Extends to f-divergences

Application: differential privacy



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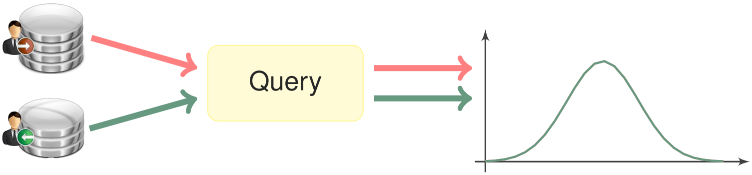

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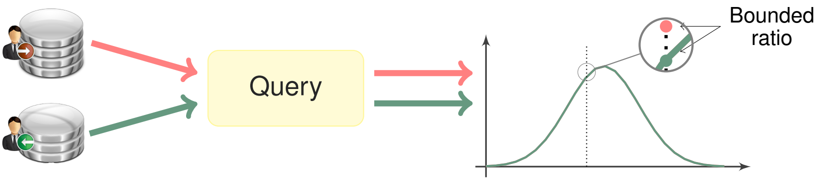

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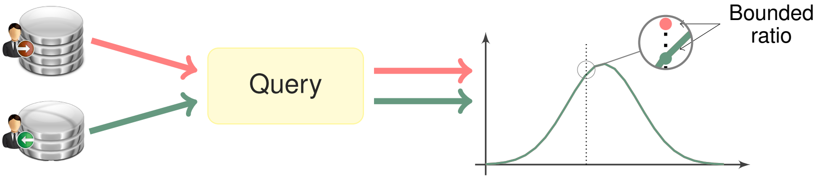

A randomized algorithm $\mathcal{K}$ is (ϵ, δ)-differentially private w.r.t. Φ iff for all databases D_1 and D_2 s.t. $\Phi(D_1, D_2)$

$$\forall S.\ \Pr[\mathcal{K}(D_1) \in S] \leq \exp(\epsilon) \cdot \Pr[\mathcal{K}(D_2) \in S] + \delta$$

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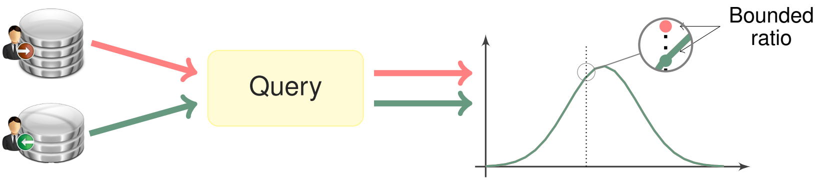

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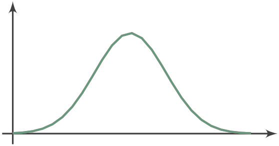
Privacy as approximate couplings

$\mathcal{K}$ is (ϵ, δ)-differentially private wrt Φ iff $\vdash_{\epsilon,\delta} \{\Phi\} \mathcal{K}_1 \sim \mathcal{K}_2 \{\equiv\}$

Differential privacy via output perturbation

Let f be k-sensitive w.r.t. Φ:

$$\Phi(a, a') \implies |f\ a - f\ a'| \leq k$$



Then $a \mapsto \mathcal{L}_\epsilon(f(a))$ is $(k \cdot \epsilon, 0)$-differentially private w.r.t. Φ

Proof principles for Laplace mechanism

Making different things look equal

$$\frac{\Phi \triangleq |e_1 - e_2| \leq k'}{\vdash_{k'\cdot\epsilon,0} \{\Phi\} y_1 \xleftarrow{\$} \mathcal{L}_\epsilon(e_1) \sim y_2 \xleftarrow{\$} \mathcal{L}_\epsilon(e_2) \{y_1 = y_2\}}$$

Making equal things look different

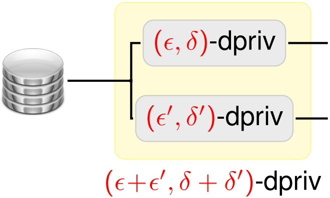
$$\frac{\Phi \triangleq e_1 = e_2}{\vdash_{k\cdot\epsilon,0} \{\Phi\} y_1 \xleftarrow{\$} \mathcal{L}_\epsilon(e_1) \sim y_2 \xleftarrow{\$} \mathcal{L}_\epsilon(e_2) \{y_1 + k = y_2\}}$$

Pointwise equality

$$\frac{\forall i.\ \vdash_{\epsilon,0} \{\Phi\} c_1 \sim c_2 \{x_1 = i \Rightarrow x_2 = i\}}{\vdash_{\epsilon,0} \{\Phi\} c_1 \sim c_2 \{x_1 = x_2\}}$$

Differential privacy by sequential composition

- If $\mathcal{K}$ is (ϵ, δ)-differentially private, and
- $\lambda a.\ \mathcal{K}'(a, b)$ is (ϵ', δ')-differentially private for every $b \in B$,
- then $\lambda a.\ \mathcal{K}'(a, \mathcal{K}(a))$ is $(\epsilon + \epsilon', \delta + \delta')$-differentially private



Beyond composition: Sparse Vector Technique

$\mathsf{SparseVector}_{bt}(a, b, M, N, d) :=$
$i \leftarrow 0; l \leftarrow []; u \xleftarrow{\$} \mathcal{L}_\epsilon(0); A \leftarrow a - u; B \leftarrow b + u;$
while $i < N$ do
  $i \leftarrow i + 1; q \leftarrow \mathcal{A}(l); S \xleftarrow{\$} \mathcal{L}_\epsilon(q(d));$
  if $(A \leq S \leq B \wedge |l| < M)$ then $l \leftarrow i :: l;$
return l

Privacy

If queries are 1-sensitive, then $(\sqrt{M}\epsilon, \delta')$-diff. private

Tools

- advanced composition
- accuracy-dependent privacy
- optimal subset coupling

Proofs as (products) programs: xpRHL

- Every pRHL derivation yields a product program
- Different derivations yield different programs
- Can be modelled by a proof system

$$\vdash \{\Phi\} c_1 \sim c_2 \{\Psi\} \rightsquigarrow c$$

Fundamental lemma of xpRHL

- $\vdash \{\Phi\} c_1 \sim c_2 \{\Psi \implies x_1 = x_2\} \rightsquigarrow c$
- $\{\Box\Phi\}\ c\ \{\Pr[\neg\Psi] \leq \epsilon\}$

implies

$$m_1\ \Phi\ m_2 \Rightarrow \left|\Pr_{(\llbracket c_1 \rrbracket\ m_1)}[E(x_1)] - \Pr_{(\llbracket c_2 \rrbracket\ m_2)}[E(x_2)]\right| \leq \epsilon$$

Dynkin's card trick (shift coupling)

$p \leftarrow s$; $l \leftarrow [p]$;
while $p < N$ **do**
    $n \xleftarrow{\$} [1, 10]$;
    $p \leftarrow p + n$;
    $l \leftarrow p :: l$;
return p

$p_1 \leftarrow s_1$; $p_2 \leftarrow s_2$;
$l_1 \leftarrow [p_1]$; $l_2 \leftarrow [p_2]$;
while $n_1 < N \vee n_2 < N$ **do**
    if $p_1 = p_2$ **then**
        $n \xleftarrow{\$} ([1, 10])$;
        $p_1 \leftarrow p_1 + n$; $p_2 \leftarrow p_2 + n$;
        $l_1 \leftarrow p_1 :: l_1$; $l_2 \leftarrow p_2 :: l_2$;
    else
        if $p_1 < p_2$ **then**
            $n_1 \xleftarrow{\$} [1, 10]$;
            $p_1 \leftarrow p_1 + n_1$;
            $l_1 \leftarrow p_1 :: l_1$;
        else
            $n_2 \xleftarrow{\$} [1, 10]$;
            $p_2 \leftarrow p_2 + n_2$;
            $l_2 \leftarrow p_2 :: l_2$;
return (p_1, p_2)

Convergence

If $s_1, s_2 \in [1, 10]$, and $N > 10$, then $\Delta(p_1^{\mathrm{final}}, p_2^{\mathrm{final}}) \leq (9/10)^{N/5-2}$

Perspectives and further directions

- Program logics for provable security and differential privacy
- Based on probabilistic couplings

Open questions

- couplings
- applications